Eco514—Game Theory
Problem Set 2: Due Thursday, October 14

Recall the following definitions: in any model $\mathcal{M} = (\Omega, (T_i, \mathbf{a}_i, p_i)_{i \in N})$, R_i is the event "Player i is rational"; $R = \bigcap_{i \in N} R_i$. Also, $B_i(E)$ is the event "Player i is certain that E is true" and $B(E) = \bigcap_{i \in N} B_i(E)$. This is as in Lecture 7.

Let me introduce the following notation for iterated mutual certainty: $B^{(0)}(E) = E$, $B^{(k)}(E) = B(B^{(k-1)}(E))$. Then the definition of B^k in Lecture 7 can be rewritten as $B^k = B^{(k)}(R)$ for $k \geq 0$.

1. A characterization of Correlated Rationalizability

Consider a game $G = (N, (A_i, u_i)_{i \in N})$. Construct a (finite) model $\mathcal{M} = (\Omega, (T_i, \mathbf{a}_i, p_i)_{i \in N})$ for G such that, for any $k \geq 0$ and for any profile of actions $a \in A$: $a \in A^{k+1}$ if and only if there exists a state $\omega \in \Omega$ such that (i) $\mathbf{a}_i(\omega) = a_i$ for all $i \in N$, and (ii) $\omega \in \bigcup_{\ell=0}^{k} B^k$.

A comment: the "if" part follows from Part (1) of Proposition 5 in Lecture 7; however, notice that Part (2) therein asserts the existence of a model that "works" for a *single* rationalizable action profile. The "only if" part of the claim I am asking you to prove instead applies to all such profiles. Also, [hint! hint!] an elegant proof of the claim does *not* rely on Proposition 5 in Lecture 7 for the "if" part.

Also, I specify that I am looking for a *finite* model because obviously the complicated "universal" model constructed in the notes for Lecture 6 will "work".

2. Mutual and Common Certainty

Consider the normal-form game with payoff uncertainty in Fig. 1:

	A	N
A	θ,θ	$\theta - 1$,0
N	0,$\theta - 1$	0,0

Figure 1: $\theta \in \{\frac{1}{2}, \frac{3}{2}\}$; $\Pr(\theta = \frac{3}{2}) = \pi$.

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(i) Begin by assuming that the probability π that $\theta = \frac{3}{2}$ is commonly known among the players. Formalize this assumption by representing the situation as a game with payoff uncertainty $G = (N, \Omega, (A_i, u_i, T_i)_{i \in N})$ as defined in Lecture 4, and compute all its Bayesian Nash equilibria (again as defined in Lectures 4 and 5).

(ii) Now fix π in the region where (N,N) and (A,A) are both Bayesian Nash equilibria of the game in which π is common knowledge. Consider the following $(k+1)$ statements:

$E = B^{(0)}(E)$ = Every player believes that $\theta = \frac{3}{2}$ with probability π;
$B^{(1)}(E)$ = Every player is certain that E is true;
...
$B^{(k)}(E)$ = Every player is certain that $B^{(k-1)}(E)$ is true.

Show that, for any $k = 0, 1, \ldots$, you can provide an alternative formulation of the game $G^k = (N, \Omega^k, (A_i, u_i^k, T_i^k)_{i \in N})$ (with the same action sets) and a pair of priors $p_i^k \in \Delta(\Omega^k)$, $i = 1, 2$, such that:
(i) there exists some state $\omega \in \Omega^k$ for which $\omega \in \bigcap_{\ell=0}^{k} B^{(\ell)}(E)$ (where of course E and certainty operators are defined in Ω^k and with respect to the priors p_i^k);
(ii) yet, there is a *unique* Bayesian Nash equilibrium of the game in which the players' priors are given by p_i^k, $i = 1, 2$, and in this equilibrium both players choose A at every state, i.e. regardless of their respective types (in the notation of Lecture 4, $\alpha_i(t_i) = \delta_{\mathrm{A}}$ for $i = 1, 2$ and all $t_i \in T_i^k$).

In words, to support (N,N) as an equilibrium, the value of π must be common certainty; mutual certainty of arbitrarily high (but finite) order is not enough.

4. From OR: 19.1, 35.2, 42.1 (important!)